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Two-Temperature Equilibration Rate for a Two-Component (e.g., Antihydrogen) Plasma in a Penning Trap

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Abstract. An expression for an equilibration time scale is developed that characterizes the rate for a group of test particles with a Maxwellian velocity distribution but different temperature from that of Maxwellian field particles to relax to the same temperature of the field particles. The expression, which can be used for any value of the Coulomb logarithm, is based on onefold integral expressions for Fokker-Planck velocity-space friction and diffusion coefficients presented elsewhere [Y. Chang and C. A. Ordonez, Phys. Rev. E **62**, 8564 (2000)]. A comparison with Spitzer's formula shows a noticeable difference when the Coulomb logarithm has values smaller than 10.

INTRODUCTION

An equilibration time scale is presented that characterizes the rate for a group of test particles with a Maxwellian velocity distribution but different temperature from that of Maxwellian field particles to relax to the same temperature of the field particles. Spitzer [1,2] formulated the equilibration time scale for a two-component plasma based on Chandrasekhar's formulas for the velocity space friction and diffusion coefficients [3,4]. Due to the approximations employed, Spitzer's formula for the equilibration time scale is limited to plasmas with large Coulomb logarithm values. The present theory is based on the binary collision model, but is not restricted in applicability by the value of the Coulomb logarithm. The reader is referred elsewhere [5,6] for more details regarding the applicability of the theory to plasmas confined in Penning traps. For example, the present theory can be used to predict the rate at which an antiproton plasma equilibrates with a low-temperature positron plasma, while both species are simultaneously confined by a nested Penning trap [7].

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THE EQUILIBRATION TIME SCALE

Consider a single test particle moving in a plasma of Maxwellian field particles. The test particle will exchange its energy with the field particles due to collisions. In a single encounter of a test particle with a field particle, the exchange of energy, ΔE, is given by

$$\Delta E = \frac{1}{2}m \left(\Delta v^2 + 2v\Delta v_{\|} \right),$$ (1)

where m is the mass of the test particle, v is its speed, Δv is the magnitude of its velocity change due to the collision, and $\Delta v_{\|}$ is the magnitude of its velocity change along its original direction. Averaging Eq. (1) over a Maxwellian velocity distribution for the field particles, the average time rate of change of the test particle energy is written as

$$\langle \Delta E \rangle = \frac{1}{2}m \left(\langle \Delta v^2 \rangle + 2v\langle \Delta v_{\|} \rangle \right).$$ (2)

In Eq. (2), $\langle \Delta v_{\|} \rangle$ and $\langle \Delta v^2 \rangle$ are the usual Fokker-Planck velocity-space friction and diffusion coefficients. Let us suppose that a group of test particles have a Maxwellian velocity distribution with temperature T, which is different from that of the field particles of temperature T_f. The equilibration time scale can be obtained simply by averaging Eq. (2) over a Maxwellian velocity distribution for the test particles. The rate of change of the test particle temperature is defined through the expression,

$$\frac{3}{2}nk\frac{dT}{dt} = \int \langle \Delta E \rangle f_M (v,T)\, dv,$$ (3)

where n is the density of the test particle species, k is Boltzmann's constant, and $f_M (v,T)$ is the Maxwellian velocity distribution with temperature T,

$$f_M (v,T) = n \left(\frac{m}{2\pi kT} \right)^{3/2} \exp \left(-\frac{m}{2kT}v^2 \right).$$ (4)

Because the test particle velocity distribution is isotropic, the integral over solid angle can be carried out and Eq. (3) can be written as

$$\frac{dT}{dt} = \frac{8\pi}{3nk} \int_0^\infty \langle \Delta E \rangle f_M (v,T)\, v^2 dv.$$ (5)

Substituting Eq. (4) into Eq. (5) and carrying out a variable change, $u = v/v_{th}$ with the thermal velocity of a field particle defined as $v_{th} = \sqrt{2kT_f/m_f}$, a non-dimensional form of Eq. (5) can be written as

$$\frac{dT}{dt} = \frac{8\zeta^{3/2}}{3\sqrt{\pi}k} \int_0^\infty \langle \Delta E \rangle \exp \left(-\zeta u^2 \right) u^2 du,$$ (6)

where $\zeta = T_f m/(T m_f)$ with m_f the mass of a field particle. The next step for the calculation of the equilibration time scale is to integrate Eq. (6). In order to integrate Eq. (6), one must substitute into Eq. (2) expressions for friction and diffusion coefficients.

SPITZER'S APPROACH TO THE EQUILIBRATION TIME SCALE

Spitzer employed Chandrasekhar's expressions for the friction and diffusion coefficients [2],

$$\langle \Delta v_{\parallel} \rangle = -\frac{4 a v_{th} \lambda G(u)}{\tau_0}, \tag{7}$$

and

$$\langle \Delta v^2 \rangle = \frac{2 (a v_{th})^2 \lambda \mathrm{erf}(u)}{\tau_0 u}, \tag{8}$$

where $a = 2\mu/m$, the reduced mass is $\mu = m m_f/(m + m_f)$, λ is the Coulomb logarithm, the time scale for single particle interactions is $\tau_0 = (n_f v_{th} \pi r_0^2)^{-1}$, n_f is the density of the field particles, the interaction radius is defined as $r_0 = Z Z_f e^2 / (8\pi\epsilon_0 k (\mu T_f / m_f))$, Z and Z_f are the charge state of a test particle and field particle, e is the unit charge, ϵ_0 is the permittivity of free space, erf is the error function, and Chandrasekhar's function is defined as $G(u) = -\frac{1}{2}\frac{d}{du}(\mathrm{erf}(u)/u)$. Substituting Eqs. (7) and (8) into Eq. (2), the integral in Eq. (6) can be carried out as

$$\frac{dT}{dt} = \frac{4m (a v_{th})^2 \lambda}{3\sqrt{\pi}\tau_0 k} \frac{\zeta^{1/2} (1 + \zeta - 2a^{-1})}{(1 + \zeta)^{3/2}}, \tag{9}$$

where the Coulomb logarithm has been approximated as being constant. Equation (9) can be rearranged into the standard form as

$$\frac{dT}{dt} = \frac{T_f - T}{\tau_{eq}^{\mathrm{Spitzer}}}, \tag{10}$$

where $\tau_{eq}^{\mathrm{Spitzer}}$ is the equilibration time scale obtained by Spitzer given by

$$\tau_{eq}^{\mathrm{Spitzer}} = \frac{3 m m_f}{8\sqrt{2\pi} n_f \lambda} \left(\frac{4\pi\epsilon_0}{Z Z_f e^2}\right)^2 \left(\frac{kT}{m} + \frac{kT_f}{m_f}\right)^{3/2}. \tag{11}$$

Spitzer's equilibration time scale applies when the Coulomb logarithm is large both because it was approximated as constant and because an approximation was employed in the process of obtaining Chandrasekhar's expressions for the friction and diffusion coefficients.

NEW APPROACH TO THE EQUILIBRATION TIME SCALE

Recently, a variable change technique [5,8–12] has been developed, which has been used for deriving the Fokker-Planck coefficients. Exact onefold integral expressions for Fokker-Planck velocity-space friction and diffusion coefficients have been obtained from

the new technique. These expressions make it possible to recalculate the equilibration time scale without placing a restriction on the value of the Coulomb logarithm. The expressions for the friction and diffusion coefficients are [5]

$$\langle \Delta v_\| \rangle = -\frac{av_{th}}{\tau_0 u^2} \int \left(\frac{\text{erf}(U) + \text{erf}(W)}{u_\delta} + \frac{\exp(-U^2) - \exp(-W^2)}{\sqrt{\pi}u_\delta^2} \right) du_\delta \qquad (12)$$

and

$$\langle \Delta v^2 \rangle = \frac{(av_{th})^2}{\tau_0 u} \int \frac{\text{erf}(U) + \text{erf}(W)}{u_\delta} du_\delta, \qquad (13)$$

where $U = u + u_\delta$, $W = u - u_\delta$, and $u_\delta = \Delta v/(av_{th})$ is the non-dimensional variable for velocity change. If we substitute Eqs. (12) and (13) into Eq. (2), Eq. (6) can be expressed as the following twofold integral:

$$\frac{dT}{dt} = \frac{4m(av_{th})^2 \zeta^{3/2}}{3\sqrt{\pi}\tau_0 k} \iint \left(\frac{\text{erf}(U) + \text{erf}(W)}{(2a^{-1} - 1)^{-1} u_\delta} - \frac{\exp(-U^2) - \exp(-W^2)}{2^{-1}\sqrt{\pi}au_\delta^2} \right) \frac{ududu_\delta}{\exp(\zeta u^2)}. \qquad (14)$$

The integral in Eq. (14) about u can be further reduced to

$$\frac{dT}{dt} = \frac{4m(av_{th})^2}{3\sqrt{\pi}\tau_0 k} \frac{\zeta^{1/2}(1 + \zeta - 2a^{-1})}{(1 + \zeta)^{3/2}} \int \exp\left(-\frac{\zeta u_\delta^2}{1 + \zeta} \right) u_\delta^{-1} du_\delta. \qquad (15)$$

Let us rewrite Eq. (15) as

$$\frac{dT}{dt} = \frac{T_f - T}{\tau_{\text{eq}}^{\text{new}}}, \qquad (16)$$

where the new equilibration time scale is obtained from Eq. (16) and Eq. (15) as

$$\tau_{\text{eq}}^{\text{new}} = \frac{3mm_f\tau}{16\mu^2} \left(\frac{1}{2} \int \exp(-x)x^{-1}dx \right)^{-1}. \qquad (17)$$

Here, $\tau = (n_f v_{av}\pi r^2)^{-1}$, where the average relative speed between two different groups of Maxwellian particles is $v_{av} = \sqrt{8kT'/(\pi\mu)}$, and the interaction radius is defined as $r = ZZ_f e^2/(8\pi\epsilon_0 kT')$. The non-dimensional interaction strength is defined as $x = H/(kT')$, in which the collision strength is defined as

$$H = \frac{\Delta p^2}{8\mu}, \qquad (18)$$

where $\Delta p = m\Delta v$ is the magnitude of the momentum transfer for a collision event, and the reduced temperature is defined as

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$$T' = \mu \left(\frac{T}{m} + \frac{T_f}{m_f} \right). \tag{19}$$

No approximations are used to arrive at Eq. (17). However the integral in Eq. (17) diverges due to the long range Coulomb interaction, and a cutoff must be imposed on the non-dimensional collision strength x. The minimum and maximum non-dimensional velocity change can be expressed in terms of the average relative speed and the Coulomb logarithm as [5]

$$u_{\delta,\min} = \frac{v_{av}}{v_{th}} e^{-\lambda} \tag{20}$$

$$u_{\delta,\max} = \frac{v_{av}}{v_{th}}. \tag{21}$$

For the two temperature system, the average relative speed is $v_{av} = 2\left((1+\zeta)/(\pi\zeta)\right)^{1/2} v_{th}$, and the integral limits for Eq. (17) become

$$x_{\min} = \frac{4}{\pi} e^{-2\lambda} \tag{22}$$

$$x_{\max} = \frac{4}{\pi}. \tag{23}$$

Substituting these into Eq. (17), we obtain the equilibration time scale

$$\tau_{eq}^{new} = \frac{3 m m_f \tau}{8 \mu^2 \Gamma\left(0, \frac{4}{\pi} e^{-2\lambda}, \frac{4}{\pi}\right)} = \frac{3 m m_f}{4\sqrt{2\pi} n_f \Gamma\left(0, \frac{4}{\pi} e^{-2\lambda}, \frac{4}{\pi}\right)} \left(\frac{4\pi\epsilon_0}{Z Z_f e^2} \right)^2 \left(\frac{kT}{m} + \frac{kT_f}{m_f} \right)^{3/2}, \tag{24}$$

where the zeroth-order incomplete gamma function $\Gamma(0, x_{\min}, x_{\max})$ is defined as

$$\Gamma(0, x_{\min}, x_{\max}) = \int_{x_{\min}}^{x_{\max}} e^{-x} x^{-1} dx. \tag{25}$$

The difference between Spitzer's approach and the present approach is characterized by the ratio

$$\frac{\tau_{eq}^{Spitzer}}{\tau_{eq}^{new}} = \frac{\Gamma\left(0, \frac{4}{\pi} e^{-2\lambda}, \frac{4}{\pi}\right)}{2\lambda}, \tag{26}$$

which is only a function of the Coulomb logarithm. With this, a comparison of Eq. (24) and Spitzer's result Eq. (11) is shown in Fig. 1. The new result Eq. (24) is generally larger than the traditional one. When the Coulomb logarithm is larger than 10, the difference is less than 5%. However, the difference can be substantial when the Coulomb logarithm is less than ten.
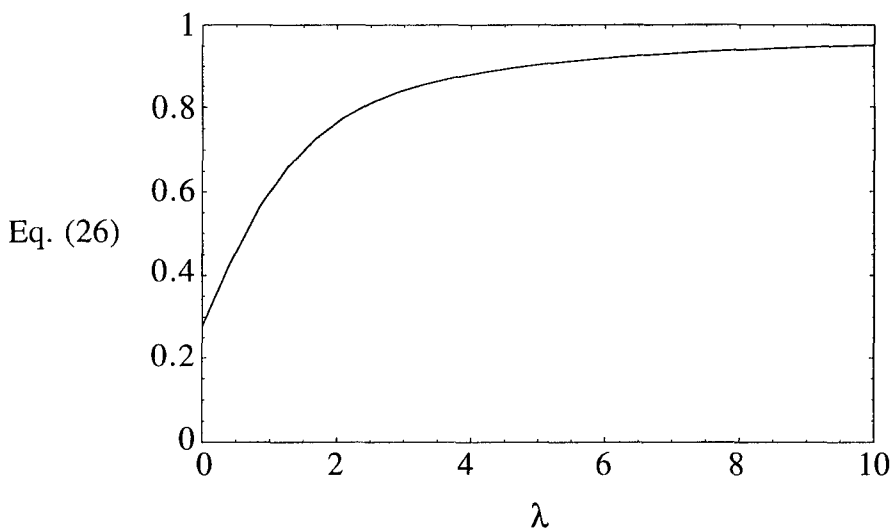
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FIGURE 1. A plot of Eq. 26.

REFERENCES

1. L. Spitzer, Monthly Notices, Roy. Astron. Soc. (London) **100**, 396 (1940).
2. L. Spitzer, *Physics of Fully Ionized Gases* (Interscience, New York, 1962), Chap. 5.
3. S. Chandrasekhar, Astrophys. J. **97**, 255 (1943).
4. S. Chandrasekhar, *Principles of Stellar Dynamics* (University of Chicago Press, Chicago, 1994), Chapter 2 and Section 5.7.
5. Y. Chang and C. A. Ordonez, Phys. Rev. E **62**, 8564 (2000).
6. J. R. Correa, Y. Chang and C. A. Ordonez, "Charge Sign Effects on the Coulomb Logarithm for a Two-Component (e.g., Antihydrogen) Plasma in a Penning Trap," *2001 Workshop on Nonneutral Plasmas, San Diego, California, July, 2001.*
7. C. A. Ordonez, Phys. Plasmas **4**, 2313 (1997).
8. Y. Chang, Phys. Fluids B **4**, 313 (1992).
9. Y. Chang, Y. Huo, and G. Yu, Phys. Fluids B **4**, 3621 (1992).
10. Y. Chang, Y. P. Huo, and J. X. Liu, Commun. Theor. Phys. **20**, 359 (1993).
11. Y. Chang and D. Li, Phys. Rev. E **53**, 3999 (1996).
12. Y. Chang and C. A. Ordonez, Phys. Plasmas **6**, 2947 (1999).